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PERIODIC SOLUTION OF THE BLACK BODY OSCILLATOR EQUATION

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Abstract: The main purpose of the present paper is to obtain an existence-uniqueness of periodic solution of the black body oscillator equation. Here we replace the classical radiation term by the relativistic form of the Dirac's radiation term proposed in a previous our paper. We use an operator presentation of the periodic problem and by means of a fixed point method we prove an existence-uniqueness of periodic solution. As a consequence, we show that Ehrenfest paradox does exist within classical physics but not in relativistic physics.

Key words: Black body oscillator equation, Periodic solution, Fixed point method, Erhrenfest paradox.

1. Introduction

It is known that in the heart of quantum mechanics the problem of radiating the black body lies (cf. [1]). The following oscillator equation

$$\ddot{x}(t)+\omega^2 x(t)-\frac{2}{3}\frac{e_0{}^2}{m_0c^3}\dddot{x}(t)=\frac{e_0}{m_0}E_x(t) \tag{1.1}$$

is assigned to the black body. Here e_0 is the charge, m_0 − the mass of the oscillator, ω − the frequency of its proper oscillations, $E_x(t)$ − one-dimensional electrical intensity of the black body radiation field (cf. for instance [1].

We consider (1.1) with respect to the velocity $x(t)=\int_0^t u(s)ds,\ x(0)=0$ and then equation (1.1) becomes

$$\dot{u}(t)+\omega^2\int_0^t u(s)ds-\frac{2}{3}\frac{e_0{}^2}{m_0c^3}\ddot{u}(t)=\frac{e_0}{m_0}E_x(t)\ . \tag{1.2}$$

Here

$$R^{rad} = -\frac{2}{3}\frac{e_0^{\ 2}}{m_0 c^3}\dddot{x}(t) \quad \left(R^{rad} = -\frac{2}{3}\frac{e_0^{\ 2}}{m_0 c^3}\ddot{u}(t) \right) \tag{1.3}$$

is the Lorentz radiation term. In [2] we have proposed a relativistic form of the radiation term following the original Dirac physical assumption (cf. [3], [4]).

The main purpose of the present paper is to replace (1.2) with its relativistic form derived in [2] (cf. [3], [4]) and to prove the existence- uniqueness of a periodic solution of the newly obtained oscillator equation.

Namely, instead of (1.3) we consider the radiation term in its one-dimensional form

$$G^{rad} = -\frac{e_0^2}{m_0 c^3 \left[1-\left(u(t)/c\right)^2\right]^{3/2}} \frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau} \ .$$

With accordance of the Dirac assumption that $\tau^{(p)ret} = \tau^{(p)adv} = \tau$, the radiation time is a small parameter τ because $\tau_0 = r_e / c \approx 2.10^{-24}\sec$ and $\tau = \tau_0\sqrt{1-u^2/c^2}$. In this manner we study (instead of (1.2)) the following oscillator equation

$$\dot{u}(t) + \omega^2 \int_0^t u(s)ds - \frac{e_0^2}{m_0\left(c^2-u^2(t)\right)^{3/2}} \frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau} = \frac{e_0}{m_0} E_x(t),\ t \geq 0 \ . \tag{1.4}$$

The existence of periodic solution we obtain by using a suitable operator (cf. [5]) whose fixed point is a periodic solution of (1.4). The conclusion at the very end of the paper are exposed.

Let us note that (1.4) is a neutral type differential equation with both retarded and advanced arguments (cf. [6]).

2. Formulation of the Periodic Problem and Preliminary Results

By $C_T^\infty[0,\infty)$ we denote the set of all infinitely differentiable *T*-periodic functions and introduce the set of functions:

$$M = \left\{ u \in C_T^\infty[0,\infty) : \left|u^I(t)\right| \leq \left|\dot{u}_0\right| + \omega U_0, \left|u^{(m)}(t)\right| \leq \omega^m U_0, \int_{pT}^{(p+1)T} u(t)dt = 0 \ , u^{(m)}(0) = 0 \ , \ t \in [pT,(p+1)T] \right\},$$

where $m = 0,2,...;\ p = 0,1,2,...;\ U_0,\ \omega,\ T,\ \mu\ ,\ \mu_0 = \mu T$ are positive constants.

Remark 2.1. We notice that the set *M* is not empty (cf. [7] and Appendix).

Let the following assumptions be fulfilled:

Assumption (E): 1) $E_x(.)\in M$, $E_x(0)=0$; 2) $n\omega_0=\omega$ (cf. [1]), $nT=T_0$ where $\omega_0=\dfrac{2\pi}{T_0}$, $\omega=\dfrac{2\pi}{T}$.

Remark 2.2. It easy to see that $E_x(t)$ is T_0 -periodic function:

$$E_x(t)=E_x(t+T)=...=E_x(t+nT)=E_x(t+T_0).$$

Assumption (C): $|u(t)|\le\overline{c}<c$ for some constant $\overline{c}$.

Assumption (U): $U_0e^{\mu T}\le\overline{c}<c.$

Following A. Sommerfeld [8] we denote by $\beta(t)=|u(t)|/c$ and by $\overline{\beta}=\overline{c}/c<1$, where $\beta(t)\le\overline{\beta}.$

Solve (1.4) with respect to $\dot{u}$ and denote the right-hand side by $U(u),$ namely

$$\dot{u}(t)=-\omega^2\int_0^t u(s)ds+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}+\frac{e_0}{m_0}E_x(t)\equiv U(u) \tag{2.1}$$

or $\dot{u}(t)=U(u),\ t\ge0.$

We need the following Lemmas. Some of them are proved in [5].

Lemma 2.1. The condition $\int\limits_{pT}^{(p+1)T}u(t)dt=0\ (p=0,1,2,3,...)$ implies that the function $x(t)=\int\limits_0^t u(s)ds$ is T - periodic, too.

Proof: $x(t+T)=\int\limits_0^{t+T}u(s)ds=\int\limits_0^t u(s)ds+\int\limits_t^{t+T}u(s)ds=\int\limits_0^t u(s)ds+\int\limits_0^T u(s)ds=\int\limits_0^t u(s)ds=x(t).$

Lemma 2.2. [5] For every $u(.)\in M$ it follows $\int\limits_{pT}^{(p+1)T}\int\limits_{pT}^{s}U(u)(\theta)d\theta ds=\int\limits_{(p+1)T}^{(p+2)T}\int\limits_{(p+1)T}^{s}U(u)(\theta)d\theta ds\ \ (p=0,1,2,...).$

Lemma 2.3 [5] The function $B(u)(t)$ belongs to $M.$

Lemma 2.4. [5] The following inequalities are satisfied:

$$|u(t)|\le\left|\int\limits_{pT}^{t}\frac{du(t_1)}{dt_1}dt_1\right|=\left|\int\limits_{pT}^{t}\left(\int\limits_{pT}^{t_1}\frac{d^2u(t_2)}{dt_2{}^2}dt_2\right)ds\right|=\left|\int\limits_{pT}^{t}\left(\int\limits_{pT}^{t_1}\left(\int\limits_{pT}^{t_2}\frac{d^3u(t_3)}{dt_3{}^3}dt_3\right)dt_2\right)dt_1\right|=...=$$

$$=\left|\int\limits_{pT}^{t}\left(\int\limits_{pT}^{t_1}\left(\int\limits_{pT}^{t_2}...\int\limits_{pT}^{t_{\kappa-1}}\frac{d^\kappa u(t_\kappa)}{dt_\kappa{}^\kappa}dt_\kappa...dt_3\right)dt_2\right)dt_1\right|\le\frac{\omega^\kappa}{\mu^\kappa}U_0e^{\mu(t-pT)}.$$

Remark 2.3. We can take $\frac{\omega^\kappa}{\mu^\kappa}$ sufficiently small for sufficiently large $\kappa \in N$, but we cannot form $\lim_{\kappa\to\infty}\frac{\omega^\kappa}{\mu^\kappa}$ because this means to define a derivative and integral of infinite order.

Lemma 2.5 The following inequalities are valid for $\tau\omega < 2$ and every $\kappa \in N$:

$$\left|\frac{\dot u(t+\tau)-\dot u(t-\tau)}{2\tau}\right| \le \frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^2 U_0}{4-(\tau\omega)^2}\ ;\ \left|\frac{u^{(m)}(t+\tau)-u^{(m)}(t-\tau)}{2\tau}\right| \le \frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^{m+1} U_0}{4-(\tau\omega)^2}.$$

Proof:

$$\dot u(t+\tau)=\dot u(t)+\tau\ddot u(t)+\frac{\tau^2}{2}\dddot u(t)+\frac{\tau^3}{3!}u^{IV}(t)+\frac{\tau^4}{4!}u^{V}(t)+\frac{\tau^5}{5!}u^{VI}(t)+\ldots;$$

$$\dot u(t-\tau)=\dot u(t)-\tau\ddot u(t)+\frac{\tau^2}{2}\dddot u(t)-\frac{\tau^3}{3!}u^{IV}(t)+\frac{\tau^4}{4!}u^{V}(t)-\frac{\tau^5}{5!}u^{VI}(t)+\ldots;$$

$$\dot u(t+\tau)-\dot u(t-\tau)=2\tau\ddot u(t)+2\frac{\tau^3}{3!}u^{IV}(t)+2\frac{\tau^5}{5!}u^{VI}(t)+\ldots=2\tau\left(u^{II}(t)+\frac{\tau^2}{3!}u^{IV}(t)+\frac{\tau^4}{5!}u^{VI}(t)+\ldots\right);$$

$$\left|\frac{\dot u(t+\tau)-\dot u(t-\tau)}{2\tau}\right| \le \left|u^{II}(t)\right|+\frac{\tau^2}{2^2}\left|u^{IV}(t)\right|+\frac{\tau^4}{2^4}\left|u^{VI}(t)\right|+\ldots \le U_0\left(\frac{\omega^\kappa}{\mu^\kappa}\omega^2+\frac{\omega^\kappa}{\mu^\kappa}\frac{\tau^2\omega^4}{2^2}+\frac{\omega^\kappa}{\mu^\kappa}\frac{\omega^6\tau^4}{2^4}+\ldots\right)=$$

$$=\frac{\omega^\kappa}{\mu^\kappa}\omega^2 U_0\left(1+\left(\frac{\tau\omega}{2}\right)^2+\left(\frac{\tau\omega}{2}\right)^4+\ldots\right)=\frac{\omega^\kappa}{\mu^\kappa}\frac{\omega^2 U_0}{1-(\tau\omega/2)^2}=\frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^2 U_0}{4-(\tau\omega)^2}.$$

Similarly

$$\left|\frac{u^{(m)}(t+\tau)-u^{(m)}(t-\tau)}{2\tau}\right| \le \left|u^{(m+1)}(t)\right|+\frac{\tau^2}{2^2}\left|u^{(m+3)}(t)\right|+\frac{\tau^4}{2^4}\left|u^{(m+5)}(t)\right|+\ldots\le$$

$$\le U_0\left(\frac{\omega^\kappa}{\mu^\kappa}\omega^{m+1}+\frac{\omega^\kappa}{\mu^\kappa}\frac{\tau^2\omega^{m+3}}{2^2}+\frac{\omega^\kappa}{\mu^\kappa}\frac{\tau^4\omega^{m+5}}{2^4}+\ldots\right)=$$

$$=\frac{\omega^\kappa}{\mu^\kappa}\omega^{m+1}U_0\left(1+\left(\frac{\tau\omega}{2}\right)^2+\left(\frac{\tau\omega}{2}\right)^4+\ldots\right)=\frac{\omega^\kappa}{\mu^\kappa}\frac{\omega^{m+1}U_0}{1-(\tau\omega/2)^2}\le\frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^{m+1}U_0}{4-(\tau\omega)^2}.$$

The Lemma 2.5 is thus proved.

Introduce the operator *B* by the formula

$$B(u)(t):=\int_{pT}^{t}U(u)(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U(u)(s)dsd\theta,\ t\in[pT,(p+1)T] \qquad (2.2)$$

We notice that $B(u)(0):=\frac{1}{2}\int_0^T U(u)(s)ds-\frac{1}{T}\int_0^T\int_0^\theta U(u)(s)dsd\theta$ and in view of (2.5) we obtain $B(u)(0)=0$. In order to avoid to prescribe an initial acceleration we replace $\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}$ by $u^{II}(t)+O(\tau^2)$ in $U(u)$, where $\tau^2\approx 10^{-48}$.

Lemma 2.6(Main Lemma). The periodic problem (2.1) has a unique solution $u\in M$ iff the operator B has a fixed point, belonging to M.

***Proof*:** Let u be a T-periodic solution of the equation

$$\frac{du(t)}{dt}=U(u)(t),\ \ t\in[0,\infty). \tag{2.3}$$

Then after integration in view of $u(pT)=0\ \ (p=0,1,2,..)$ we obtain

$$u(t)=\int_{pT}^{t}U(u)(s)ds\ \Rightarrow\ 0=u((p+1)T)=\int_{pT}^{(p+1)T}U(u)(s)ds\ \Rightarrow\ \int_{pT}^{(p+1)T}U(u)(s)ds=0.$$

Therefore the operator B becomes

$$B(p)(t)=\int_{pT}^{t}U(u)(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U(u)(s)ds. \tag{2.4}$$

Since the equation (2.3) has a T-periodic solution,then changing the order of integration and taking into account $u(pT)=0$ and $U(u)(.),\ u(.)\in M$ we obtain

$$\int_{pT}^{(p+1)T}\int_{pT}^{t}U(u)(s)dsdt=\int_{pT}^{(p+1)T}\left[(p+1)T-s\right]U(u)(s)ds=(p+1)T\int_{pT}^{(p+1)T}U(u)(s)ds-\int_{pT}^{(p+1)T}sU(u)(s)ds=-\int_{pT}^{(p+1)T}sU(u)(s)ds.$$

But $\frac{du(t)}{dt}=U(u)(t)$ and therefore

$$\int_{pT}^{(p+1)T}s\frac{du(s)}{ds}ds=\int_{pT}^{(p+1T}sd\left(u(s)\right)=\left[(p+1)T\ u((p+1)T)-pT\,u(pT)\right]-\int_{pT}^{(p+1T}u(s)ds=0.$$

Consequently $\int_{pT}^{(p+1)T}\int_{pT}^{t}U(u)(s)dsdt=0.$ Therefore the equality (2.4) becomes

$$B(u)(t)=\int_{pT}^{t}U(u)(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U(u)(s)dsd\theta.$$

We have proved that if (2.3) has a T-periodic solution then B has a fixed point.

Conversely, let $u(.)\in M$ be a fixed point of B, that is, $u=B(u)$. Therefore $u(pT)=B(u)(pT)$ or

$$0=u(pT)=B(u)(pT)=\int_{pT}^{pT}U(u)(s)ds-\left(\frac{pT-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{s}U(u)(\theta)d\theta ds=$$

$$=\frac{1}{2}\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{s}U(u)(\theta)d\theta ds.$$

It follows

$$\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{s}U(u)(\theta)d\theta ds=\frac{1}{2}\int_{pT}^{(p+1)T}U(u)(s)ds. \tag{2.5}$$

We show that $\int_{pT}^{(p+1)T}U(u)(s)ds=0.$ Indeed, in view of Lemma 2.5we get

$$\left|\int_{pT}^{(p+1)T}U(u)ds\right|\le\left|\omega^2\int_{pT}^{(p+1)T}\int_{pT}^{t}u(s)dsdt\right|+\frac{e_0^2}{m_0}\int_{pT}^{(p+1)T}\frac{e^{\mu(t-pT)}}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}dt\frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^2U_0}{4-(\tau\omega)^2}+\left|\frac{e_0}{m_0}\int_{pT}^{(p+1)T}E_x(t)dt\right|\le$$

$$\le\left|\omega^2\int_{pT}^{(p+1)T}\int_{pT}^{t}\int_{pT}^{t_1}\left(|\dot{u}_0|+\int_{pT}^{t_2}u^{II}(t_3)\right)dt_2dt_1dt\right|+\frac{e_0^2}{m_0}\frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^2U_0}{4-(\tau\omega)^2}\frac{1}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}\int_{pT}^{(p+1)T}e^{\mu(t-pT)}dt+\frac{e_0}{m_0}\frac{\omega^{\kappa+1}}{\mu^{\kappa+1}}U_0\frac{e^{\mu T}-1}{\mu}\le$$

$$\le\frac{e^{\mu T}-1}{\mu}\frac{|\dot{u}_0|\omega^2}{\mu^3}+\frac{e^{\mu T}-1}{\mu}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{m_0}\right)\omega U_0\ . \tag{2.6}$$

Indeed, if $\delta=\left|\int_{pT}^{(p+1)T}U(u)(s)ds\right|>0$ and $\mu>\omega>0$ we have noticed that the above inequality might be violated. Therefore the operator

$$B(u)(t)=\int_{pT}^{t}U(u)(s)ds-\left(\frac{t-pT}{T}-\frac{1}{2}\right)\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{T}^{(p+1)T}\int_{pT}^{\theta}U(u)(s)dsd\theta$$

in view of (2.5) becomes $B(u)(t)=\int_{pT}^{t}U(u)(s)ds\Leftrightarrow u(t)=\int_{pT}^{t}U(u)(s)ds$. Differentiating the last equality we obtain that the fixed point of the operator B is a T-periodic solution of $\dot{u}(t)=U(u)(t)$.

Lemma 2.7 is thus proved.

3. Existence-Uniqueness of Periodic Solution of the Oscillator Equation

We introduce the family of pseudometrics

$$\rho_{(p,m)}\left(u,\bar{u}\right)=\sup\left\{e^{-\mu(t-pT_0)}\omega^{-m}\left|\frac{d^m u(t)}{dt^l}-\frac{d^m\bar{u}(t)}{dt^l}\right|:t\in[pT_0,(p+1)T_0]\right\},(p=0,1,2,...;\ m=0,1,2)\ .$$

One can notice that

$$\rho_{(p,m)}\left(u,\bar{u}\right)=\sup\left\{e^{-\mu(t-pT)}\omega^{-m}\left|\frac{d^m u(t)}{dt^m}-\frac{d^m\bar{u}(t)}{dt^m}\right|:t\in[pT,(p+1)T]\right\}\leq$$

$$\leq\sup\left\{e^{-\mu(t-pT)}\omega^{-m}\left(\left|\frac{d^m u(t)}{dt^m}\right|+\left|\frac{d^m\bar{u}(t)}{dt^m}\right|\right):t\in[pT,(p+1)T]\right\}\leq$$

$$\leq\sup\left\{e^{-\mu(t-pT)}\omega^{-m}\omega^m e^{\mu(t-pT)}2U_0:t\in[pT,(p+1)T]\right\}=2U_0<\infty(p=0,1,2,...). \qquad (3.1)$$

Theorem 3.1 (Main result). Let the assumptions **(E), (C), (U)** be fulfilled. The numbers are $\mu>\omega,\ \kappa\in N$ sufficiently large. Then there is a unique *T*-periodic solution $u(t)$ of (2.3) for $t\geq 0$.

Proof: With accordance of the Main Lemma 2.6 we have to prove that the operator defined by (2.2) possesses a unique fixed point which means that our periodic problem has a unique solution.

We have already established that the operator functions $B(u)(t)$ are *T*-periodic ones. Recall that

$$U(u)(t)=-\omega^2\int_0^t u(s)ds+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\ddot{u}(t)+\frac{e_0}{m_0}E_x(t)$$

we have

$$B(u)(0)=\frac{1}{2}\int_{pT}^{(p+1)T}U(u)(s)ds-\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{\theta}U(u)(s)dsd\theta=0\,\mathbf{;}$$

$$\frac{dB(u)(t)}{dt}=U(u)(t)-\frac{1}{T}\int_0^T U(u)(s)ds\ \Rightarrow\frac{dB(u)(0)}{dt}=U(u)(0)-\frac{1}{T}\int_0^T U(u)(s)ds=-\frac{1}{T}\int_0^T U(u)(s)ds\ ;$$

$$\left|\frac{dB(u)(0)}{dt}\right|\leq\frac{1}{T}\left|\int_0^T U(u)(s)ds\ \right|\leq$$

$$\leq\frac{e^{\mu T}-1}{\mu T}\frac{\left|\dot{u}_0\right|\omega^2}{\mu^3}+\frac{e^{\mu T}-1}{\mu T}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0}\frac{4\omega}{4-\left(\tau\omega\right)^2}\frac{1}{c^3\left(1-\overline{\beta}^2\right)^{3/2}}+\frac{e_0}{m_0}\right)\omega U_0\leq\left(\left|\dot{u}_0\right|+\omega U_0\right)e^{\mu t}\ ;$$

$$\frac{d^2B(u)(t)}{dt^2}=\frac{dU(u)(t)}{dt}=-\omega^2u(t)+\frac{e_0^2}{m_0c^3}\left(-\frac{3}{2}\right)\frac{-2\beta(t)\dot{\beta}(t)}{\left(1-\beta^2(t)\right)^{5/2}}\ddot{u}(t)+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\dddot{u}(t)+\frac{e_0}{m_0}\dot{E}_x(t)\ ;$$

$$\frac{d^2B(u)(0)}{dt^2}=0-\omega^2u(0)+\frac{e_0^2}{m_0c^3}\left(-\frac{3}{2}\right)\frac{-2\beta(0)\dot{\beta}(0)}{\left(1-\beta^2(0)\right)^{5/2}}\ddot{u}(0)+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\ddot{u}(0)+\frac{e_0}{m_0}\dot{E}_x(0)=0$$

and so, on.

In view of $\int_{pT}^{(p+1)T}\left(\frac{t-pT}{T}-\frac{1}{2}\right)dt=0$ we obtain

$$\int_{pT}^{(p+1)T}B(u)(u)(t)dt=\int_{pT}^{(p+1)T}\int_{pT}^{t}U(u)(s)dsdt-\int_{pT}^{(p+1)T}\left(\frac{t-pT}{T}-\frac{1}{2}\right)dt\int_{pT}^{(p+1)T}U(u)(s)ds-T\frac{1}{T}\int_{pT}^{(p+1)T}\int_{pT}^{t}U(u)(s)dsdt=0\ ,$$

that is, $B(u)(.)\in M$.

The following inequalities are valid ($nT=T_0$):

$$|B(u)(t)|\le\left|\int_{pT}^{t}U(u)(s)ds\right|+\left|\int_{pT}^{(p+1)T}U(u)(s)ds\right|\le$$

$$\le\omega^2\left|\int_{pT}^{t}\int_{pT}^{s}u(\xi)d\xi ds\right|+\frac{e_0^2}{m_0}\int_{pT}^{t}\frac{e^{\mu(s-pT)}}{c^3\left(1-\beta^2(s)\right)^{3/2}}ds\frac{\omega^\kappa}{\mu^\kappa}\frac{4\omega^2U_0}{4-(\tau\omega)^2}+\left|\frac{e_0}{m_0}\int_{pT}^{t}E_x(s)ds\right|+$$

$$+\frac{e^{\mu T}-1}{\mu}\frac{|\dot{u}_0|\omega^2}{\mu^3}+\frac{e^{\mu T}-1}{\mu}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{m_0}\right)\omega U_0\le$$

$$\le e^{\mu(t-pT)}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega^2}{\mu^2}+\frac{e_0^2}{m_0}\frac{1}{\mu c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4\omega^2}{4-(\tau\omega)^2}+\frac{\omega}{\mu}\frac{|e_0|}{m_0}\right)U_0+$$

$$+\frac{e^{\mu T}-1}{\mu}\frac{|\dot{u}_0|\omega^2}{\mu^3}+\left(e^{\mu T}-1\right)\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu^2}+\frac{e_0^2}{m_0}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{\mu c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{\mu m_0}\right)\omega U_0\le$$

$$\le e^{\mu(t-pT)}\left(\frac{e^{\mu T}-1}{\mu}\frac{|\dot{u}_0|\omega^2}{\mu^3}+e^{\mu T}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu^2}+\frac{e_0^2}{m_0}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{\mu c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{\mu m_0}\right)\omega U_0\right]\le e^{\mu(t-pT)}U_0\ .$$

In view of (2.1) for the first derivative of $B(u)(t)$ we have

$$\left|\dot{B}(u)(t)\right|\le|U(u)(t)|+\left|\frac{1}{T}\int_{pT}^{(p+1)T}U(u)(s)ds\right|\le$$

$$\le\omega^2\left|\int_0^t u(s)ds\right|+\frac{e_0^2}{m_0c^3\left(1-\bar{\beta}^2\right)^{3/2}}|\ddot{u}(t)|+\frac{|e_0|}{m_0}|E_x(t)|+$$

$$+\frac{e^{\mu T}-1}{\mu T}\frac{|\dot{u}_0|\omega^2}{\mu^3}+\frac{e^{\mu T}-1}{\mu T}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{m_0}\right)\omega U_0\le$$

$$\le e^{\mu(t-pT)}\left[\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0c^3}\frac{\omega}{\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{\mu m_0}\right)\omega U_0+\right.$$

$$\left.+\frac{e^{\mu T}-1}{\mu T}\frac{|\dot{u}_0|\omega^2}{\mu^3}+\frac{e^{\mu T}-1}{\mu T}\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega}{\mu}+\frac{e_0^2}{m_0c^3}\frac{4\omega}{4-(\tau\omega)^2}\frac{1}{\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{|e_0|}{m_0}\right)\omega U_0\right]\le\omega U_0e^{\mu(t-pT)} .$$

Since

$$U(u)(t)=-\omega^2\int_0^t u(s)ds+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\ddot{u}(t)+\frac{e_0}{m_0}E_x(t) ;$$

$$\frac{dU(u)(t)}{dt}=-\omega^2u(t)+\frac{e_0^2u^{III}(t)}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}+\frac{3e_0^2u(t)u^I(t)}{m_0c^5\left(1-\beta^2(t)\right)^{5/2}}u^{II}(t)+\frac{e_0}{m_0}E^I{}_x(t)$$

for the second derivative of $B(u)(t)$ we get

$$\left|B^{II}(u)(t)\right|\le\left|\frac{dU(u)(t)}{dt}\right|\le e^{\mu(t-pT)}\frac{\omega^\kappa}{\mu^\kappa}\left(1+\frac{e_0^2\omega}{m_0c^3\left(1-\bar{\beta}^2\right)^{3/2}}+\frac{3e_0^2\left(|\dot{u}_0|+\omega U_0\right)}{m_0c^4\left(1-\bar{\beta}^2\right)^{5/2}}+\frac{|e_0|}{m_0}\frac{1}{\mu^2}\right)\omega^2U_0\le\omega^2U_0e^{\mu(t-pT)} .$$

For the higher order derivatives, we proceed in a similar way.

Consequently, *B* maps the set M into itself.

It remains to show that *B* is a contractive operator.

Consider the following index set A consisting of all (p,m). The set M turns out into a uniform space with a saturated family of pseudo-metrics $\rho_{(p,m)}\left(u,\bar{u}\right)$ where *p* and *m* run over all non-negative integers.

To estimate the difference $\left|B(u)(t)-\bar{B}(u)(t)\right|\le\int_{pT}^{t}\left|U(u)(s)-\bar{U}(u)(s)\right|ds+\frac{1}{T}\int_{pT}^{(p+1)T}\left|U(u)(s)-\bar{U}(u)(s)\right|ds$

we notice $U(u)=-\omega^2\int_0^t u(s)ds+\frac{e_0^2}{m_0c^3\left(1-\beta^2(t)\right)^{3/2}}\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}+\frac{e_0}{m_0}E_x(t).$ Then

$$\int_{pT}^{t}\left|U(u)(s)-U(\bar{u})(s)\right|ds\le$$

$$\leq \omega^2 \left| \int_{pT}^{t} \int_{pT}^{s} \left(u(\theta) - \bar{u}(\theta)\right) d\theta ds \right| + \frac{e_0{}^2}{m_0} \frac{1}{c^3\left(1-\bar{\beta}^2\right)^{3/2}} \left| \int_{pT}^{t} \left(u^{II}(s) - \bar{u}^{II}(s)\right) ds \right| \approx$$

$$\approx e^{\mu(t-pT)} \frac{\omega^2}{\mu^2} \rho_{(p,0)}(u,\bar{u}) + \frac{e_0{}^2}{m_0 c^3\left(1-\bar{\beta}^2\right)^{3/2}} \left| u^I(t) - \bar{u}^I(t) \right| \leq e^{\mu(t-pT)} \frac{\omega^\kappa}{\mu^\kappa} \left(\frac{\omega^2}{\mu^2} \rho_{(p,\kappa)}(u,\bar{u}) + \frac{e_0{}^2 \omega}{m_0 c^3\left(1-\bar{\beta}^2\right)^{3/2}} \rho_{(p,\kappa+1)}(u,\bar{u}) \right) ;$$

$$\frac{1}{T} \int_{pT}^{(p+1)T} \left| U(u)(s) - \bar{U}(u)(s) \right| ds \leq e^{\mu(t-pT)} \frac{e^{\mu T} - 1}{\mu T} \frac{\omega^\kappa}{\mu^\kappa} \left(\frac{\omega^2}{\mu^2} + \frac{e_0{}^2 \omega}{m_0 c^3\left(1-\bar{\beta}^2\right)^{3/2}} \right) \left(\rho_{(p,\kappa)}(u,\bar{u}) + \rho_{(p,\kappa+1)}(u,\bar{u}) \right).$$

Therefore $\rho_{(p,0)}\left(B(u), B(\bar{u})\right) \leq \frac{e^{\mu T} - 1}{\mu T} \frac{\omega^\kappa}{\mu^\kappa} \left(\frac{\omega^2}{\mu^2} + \frac{e_0{}^2 \omega}{m_0 c^3\left(1-\bar{\beta}^2\right)^{3/2}} \right) \left(\rho_{(p,\kappa)}(u,\bar{u}) + \rho_{(p,\kappa+1)}(u,\bar{u}) \right).$

But $\left| \frac{d^\kappa u(t)}{dt^\kappa} - \frac{d^\kappa \bar{u}(t)}{dt^\kappa} \right| = \left| \int_0^t \left(\frac{d^{\kappa+1} u(s)}{ds^{\kappa+1}} - \frac{d^{\kappa+1} \bar{u}(s)}{ds^{\kappa+1}} \right) ds \right| \leq \omega^{\kappa+1} \rho_{(p,\kappa+1)}(u,\bar{u}) \int_0^t e^{\mu s} ds = \frac{\omega^{\kappa+1} \rho_{(p,\kappa+1)}(u,\bar{u})}{\mu} e^{\mu t}$.

Therefore $\frac{e^{-\mu t}}{\omega^\kappa} \left| \frac{d^\kappa u(t)}{dt^\kappa} - \frac{d^\kappa \bar{u}(t)}{dt^\kappa} \right| \leq \frac{\omega \rho_{(p,\kappa+1)}(u,\bar{u})}{\mu} \Rightarrow \rho_{(p,\kappa)}(u,\bar{u}) \leq \frac{\omega}{\mu} \rho_{(p,\kappa+1)}(u,\bar{u}).$

Then $\rho_{(p,0)}\left(B(u), B(\bar{u})\right) \leq \frac{e^{\mu T} - 1}{\mu T} \frac{\omega^\kappa}{\mu^\kappa} \left(\frac{\omega^2}{\mu^2} + \frac{e_0{}^2 \omega}{m_0 c^3\left(1-\bar{\beta}^2\right)^{3/2}} \right) \left(\frac{\omega}{\mu} + 1 \right) \rho_{(p,\kappa+1)}(u,\bar{u}).$

Define the map $j:(p,l) \to (p, l+\kappa+1).$ In view of (3.1) the uniform space is *j*-bounded (cf. [9]).

Therefore, the operator *B* is contractive in the sense defined in [9] and the unique fixed point of *B* is a *T*-periodic solution of the oscillator equation.

Theorem 3.1 is thus proved.

4. Conclusion

Let us consider the inequalities implying the existence-uniqueness of *T*-periodic solution.

Since $u(t), \dot{u}(t), \ \ddot{u}(t), \ \int_0^t u(s) ds$ are *T*-periodic functions in view of

$$E_x(t) = \frac{m_0}{e_0} \dot{u}(t) + \frac{m_0}{e_0} \omega^2 \int_0^t u(s) ds - \frac{e_0}{c^3\left(1-\beta^2(t)\right)^{3/2}} \frac{\dot{u}(t+\tau) - \dot{u}(t-\tau)}{2\tau}$$

for $n\omega_0 = \omega, \ nT = T_0$ we get

$$E=\frac{3}{4\pi}\frac{1}{T_0}\int_0^{T_0}E_x^2(t)dt=\frac{3}{4\pi}\frac{1}{T_0}\int_0^{T_0}\left(\frac{m_0}{e_0}\dot{u}(t)+\frac{m_0}{e_0}\omega^2\int_0^t u(s)ds-\frac{e_0}{c^3\left(1-\beta^2(t)\right)^{3/2}}\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}\right)^2 dt=$$

$$=\frac{3}{4\pi}\frac{1}{nT}\int_0^{nT}\left(\frac{m_0}{e_0}\dot{u}(t)+\frac{m_0}{e_0}\omega^2\int_0^t\int_0^s\dot{u}(\xi)d\xi ds+\frac{e_0}{c^3\left(1-\beta^2(t)\right)^{3/2}}\frac{\dot{u}(t+\tau)-\dot{u}(t-\tau)}{2\tau}\right)^2 dt.$$

In view of Lemma 2.5 and $\varphi(\omega)=\frac{\omega^2}{4-\tau^2\omega^2}\Rightarrow\ \varphi^I(\omega)=\frac{2\omega\left(4-\tau^2\omega^2\right)+\omega^2.2\tau^2\omega}{\left(4-(\tau\omega)\right)^4}=\frac{8\omega}{\left(4-(\tau\omega)\right)^4}>0.$

We notice $\omega<\frac{2}{\tau}=\frac{2}{\tau_0\sqrt{1-\beta^2}}\le\frac{2}{\tau_0\sqrt{1-\bar{\beta}^2}}=\bar{\omega}$ and therefore

$$\frac{|e_0|}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4U_0\omega^2}{4-(\tau\omega)^2}\le\frac{|e_0|}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4\bar{\omega}^2U_0}{4-(\tau\bar{\omega})^2}.$$

Consequently

$$E\le\frac{3}{4\pi T}\int_0^T\left(\frac{m_0\omega U_0}{|e_0|}+\frac{m_0}{e_0}\frac{\omega^3U_0e^{\mu T}}{\mu^2}+\frac{|e_0|}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4\omega^2U_0}{4-(\tau\omega)^2}\right)^2 dt\le\frac{3}{4\pi}\left(\frac{m_0\bar{\omega}U_0}{|e_0|}+\frac{m_0}{|e_0|}\frac{\bar{\omega}^3c}{\mu^2}+\frac{|e_0|}{c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4\bar{\omega}^2U_0}{4-(\tau\bar{\omega})^2}\right)^2<\infty.$$

We check the inequalities of the Main Theorem. Let us choose $\bar{c}/c=0{,}999999,$ then

$\left(1-\bar{\beta}^2\right)^{1,5}=\left(1-0{,}999999^2\right)^{1,5}\approx\left(1{,}4.10^{-3}\right)^3\approx2{,}74.10^{-9}$; $\left(1-\bar{\beta}^2\right)^{2,5}=\left(1-0{,}999999^2\right)^{2,5}\approx\left(1{,}4.10^{-3}\right)^5\approx5{,}4.10^{-15}$;

$$\bar{\omega}=\frac{2}{\tau_0\sqrt{1-(\bar{c}/c)^2}}=\frac{2}{10^{-24}.1{,}4.10^{-3}}\approx10^{27}\Rightarrow\ \bar{\nu}=\frac{\bar{\omega}}{2\pi}=\frac{2}{2\pi.\tau_0\sqrt{1-(\bar{c}/c)^2}}=\frac{1}{3{,}14.10^{-24}.1{,}4.10^{-3}}\approx10^{26}\,\text{Hz};$$

$\tau\sqrt{1-\beta^2}\,\omega\le1{,}4<2;\ \ \frac{e_0^2}{m_0}=2{,}8.10^{-8};\ \ \frac{|e_0|}{m_0}=\frac{1{,}6.10^{-13}}{9.10^{-31}}\approx1{,}8.10^{17}.$

For the Roentgen rays with $\lambda_R=10^{-11}\,\text{m}$ we have $\nu_R=c/\lambda_R=3.10^8/10^{-11}=3.10^{19}\,\text{Hz},$

$\omega_R=2\pi\nu_R\approx6{,}28.3.10^{19}\approx1{,}9.10^{20}\,\text{Hz},\qquad T_R=\frac{2\pi}{\omega_R}=\frac{2\pi}{1{,}9.10^{20}}\approx3{,}3.10^{-20}\,\text{sec}.$ We take $\mu=2.10^{20}>\omega_R=1{,}9.10^{20}.$

Then $\mu T_R=6{,}27$ and $\frac{\omega_R}{\mu}=\frac{2\pi\nu_R}{\mu}=\frac{1{,}9.10^{20}}{2.10^{20}}\approx0{,}95.$ Therefore

$$2\left(e^{\mu T}-1\right)\frac{\omega^\kappa}{\mu^\kappa}\left(\frac{\omega^2}{\mu^2}+\frac{e_0^2}{\mu m_0c^3\left(1-\bar{\beta}^2\right)^{3/2}}\frac{4\omega^2}{4-(\tau\omega)^2}+\frac{|e_0|}{m_0}\frac{1}{\mu}\right)\le1\Rightarrow$$

$$2\left(e^{6,27}-1\right)\frac{\omega^{\kappa}}{\mu^{\kappa}}\left(\frac{\left(1,9.10^{20}\right)^2 3,3^2}{10^{40}}+\frac{3,3}{10^{20}}\frac{4.2,8.10^{-8}}{9.10^{24}.2,74.10^{-9}}\frac{4\left(1,9.10^{20}\right)^2}{4-\left(1,4\right)^2}+\frac{1,8.10^{17}.3,3}{10^{20}}\right)\approx\left(0,95\right)^{\kappa}.528,48\leq 1.$$

The last inequality is satisfied for sufficiently large $\kappa\in N$.

Then the question of convergence of the integral $E=\int_0^{\infty}\rho(\omega)d\omega$ with density $\rho(\omega)=\frac{kT}{\pi^2c^3}\omega^2$ (Rayleigh–Jeans law) falls away, because we should consider the above integral on the finite interval $\left[0,\frac{2}{\tau_0\sqrt{1-\left(\bar{c}/c\right)^2}}\right]$, that is, $\int_0^{2/\tau_0\sqrt{1-\left(\bar{c}/c\right)^2}}\omega^2 d\omega=\int_0^{10^{27}}\omega^2 d\omega$. Obviously, $\lim_{\bar{c}\to c}\frac{2}{\tau_0\sqrt{1-\left(\bar{c}/c\right)^2}}=\infty.$

We note that in the classical (Newtonian) physics we have $c=\infty$, but there is no motion with $c=\infty$. In a similar way we can not reach *c* and assume that the velocities satisfy the condition $|u(t)|\leq\bar{c}<c<\infty$.

We have obtained that the energy has a finite value. A natural consequence arises − it is not necessary to assume that the energy takes only discrete values to avoid the Ehrenfest paradox.

Appendix

Here we give an example of functions belonging to the set *M*:

$$u(t)=\begin{cases}e^{-T^2/t(2T-t)}, & t\in(0,2T)\\ 0, & t\,\overline{\in}\,(0,2T)\end{cases};\quad u^{I}(t)=e^{-T^2/t(2T-t)}\left(-\frac{T^2}{t(2T-t)}\right)^{I}=2T^2e^{-T^2/t(2T-t)}\frac{(T-t)}{\left(2Tt-t^2\right)^2}.$$

Therefore $|u(t)|\leq e^{-T^2/t(2T-t)}=e^{-1}$.

For $t\in[0,2T]$ we have $2|T-t|\leq 2T$ and in view of $\frac{1}{T}=\frac{\omega}{2\pi}\Rightarrow$

$$|u^{I}(t)|\leq 2T^3e^{-T^2/t(2T-t)}\frac{1}{\left(2Tt-t^2\right)^2}\leq\frac{2T^3e^{-1}}{T^4}=\frac{1}{e\pi}\omega\approx 0,117\omega.$$

We have

$$u^{II}(t)=2T^2\frac{d}{dt}\left[e^{-T^2/t(2T-t)}\frac{(T-t)}{\left(2Tt-t^2\right)^2}\right]=2T^2e^{-T^2/t(2T-t)}\frac{2T^2(T-t)^2-\left(2Tt-t^2\right)^2-4(T-t)^2\left(2Tt-t^2\right)}{\left(2Tt-t^2\right)^4}\leq$$

$$\leq 4T^4e^{-T^2/t(2T-t)}\frac{(T-t)^2}{\left(2Tt-t^2\right)^4}\leq 4T^6e^{-T^2/t(2T-t)}\frac{1}{\left(2Tt-t^2\right)^4}.$$

Since $\frac{d}{dt}\left(e^{-T^2/t(2T-t)}\frac{1}{\left(2Tt-t^2\right)^4}\right)=e^{-T^2/t(2T-t)}(T-t)\left(2T^2\frac{1}{\left(2Tt-t^2\right)^2}\frac{1}{\left(2Tt-t^2\right)^4}-\frac{8}{\left(2Tt-t^2\right)^5}\right)=0$ for

$t=T$ we have

$$\left|u^{II}(t)\right|\le 4T^4e^{-T^2/t(2T-t)}\frac{(T-t)^2}{\left(2Tt-t^2\right)^4}\le 4T^6e^{-T^2/t(2T-t)}\frac{1}{\left(2Tt-t^2\right)^4}\le e^{-1}\frac{4T^6}{\left(T^2\right)^4}=\frac{2^2\omega^2}{e\left(2\pi\right)^2}=\frac{\omega^2}{e\left(\pi\right)^2}\approx 0,04.\omega^2.$$

Further on we obtain

$$u^{III}(t)=2T^2\left[\left(\frac{d^2}{dt^2}e^{-\frac{T^2}{t(2T-t)}}\right)\frac{(T-t)}{\left(2Tt-t^2\right)^2}+2\left(\frac{d}{dt}e^{-\frac{T^2}{t(2T-t)}}\right)\left(\frac{d}{dt}\frac{(T-t)}{\left(2Tt-t^2\right)^2}\right)+e^{-\frac{T^2}{t(2T-t)}}\left(\frac{d^2}{dt^2}\frac{(T-t)}{\left(2Tt-t^2\right)^2}\right)\right]=$$

$$=2T^2e^{-\frac{T^2}{t(2T-t)}}(T-t)\left[2T^2\frac{2T^2(T-t)^2-\left(2Tt-t^2\right)^2-4(T-t)^2\left(2Tt-t^2\right)}{\left(2Tt-t^2\right)^4}\frac{1}{\left(2Tt-t^2\right)^2}+\right.$$

$$\left.+\frac{2T^2}{\left(2Tt-t^2\right)^2}\frac{-\left(2Tt-t^2\right)^2-4(T-t)^2\left(2Tt-t^2\right)}{\left(2Tt-t^2\right)^4}-+\frac{-6\left(2Tt-t^2\right)^3-6(4T^2-6tT+3t^2)\left(2Tt-t^2\right)^2}{\left(2Tt-t^2\right)^6}\right]\le$$

$$\le 2T^3e^{-\frac{T^2}{t(2T-t)}}\frac{4T^4(T-t)^2}{\left(2Tt-t^2\right)^6}\le\frac{8T^9}{e\left(T^2\right)^6}=\frac{2^3}{e(2\pi)^3}\omega^3$$

$$\left|\frac{d^mu(t)}{dt^m}\right|\le\frac{2^m}{e(2\pi)^m}\omega^m$$

.

Finally, to obtain $\int_0^{4T}u(t)dt=0$ we define $u(t)=\begin{cases}e^{-T^2/t(2T-t)} & ,\ t\in(0,2T)\\ -e^{-T^2/t(2T-t)} & ,\ t\in(2T,4T)\\ 0 & ,t\,\overline{\in}(0,4T)\end{cases}$.

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