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Optimal Investment Strategy for an Investor under Modified Constant Elasticity of Variance (M-CEV) and Ornstein – Uhlenbeck Models through Power Utility maximization

By

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ABSTRACT

This work studied the optimal investment strategy for an investor under Modified Constant Elasticity of Variance (M-CEV) and Ornstein-Uhenbeck models. The stock price is assumed to be governed by the M-CEV model and the goal is to find the optimal investment strategy where the investor has a power utility preference when the Brownian motions do not and do correlate. The application of maximum principle of dynamic programming helped us to obtain the required Hamilton-Jacobi-Bellman (HJB) equation. The method elimination of variable dependency was applied to transform the second order partial differential equation to an ordinary differential equation from which the close form solution of the optimal investment strategy was obtained. It is found that the investor's optimal strategy when the Brownian motions correlate is less than the investor's optimal investment strategy when the Brownian motions do not correlate by a fraction of the total wealth.

Keywords: Optimal Investment Strategy, Investor, Modified Constant Elasticity of Variance (M-CEV), Ornstein Uhlenbeck Model, Power Utility maximization

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INTRODUCTION

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The idea of an "optimal portfolio" comes from the modern portfolio theory (MPT), it is a theory on how risk averse investors can construct portfolios to optimize the expected return based on a given level of market risk, emphasizing that risk is an inherent part of higher reward. According to this theory, it's possible to construct an efficient frontier of optimal portfolios offering the maximum possible expected return for a given level of risk. This theory was pioneered by Harry Markowitz in his paper "portfolio selection", published in 1952 by the Journal of Finance.

Modern portfolio theory argues that an investment's risk and return characteristics should not be viewed alone, but should be evaluated by how the investment affects the overall portfolio's risk and return. This means that individual investors should determine how much risk they are willing to take, allocate or diversify their portfolios according to the results observed.

An individual investor can determine how much volatility he or she is willing to maintain in his/her other portfolio by picking another point which has along the so called efficient frontier. Doing so will provide maximum return for the amount of risk that the investor has decided to accept. Of course, optimizing a portfolio in practical terms is quite difficult and cannot be done easily. Today, there are computer programs and services which are devoted to determining optimal portfolios. The way that they accomplish this is by estimating different expected returns thousands of times over for each amount of risk.

In this work we solved an optimization problem assuming the modified constant elasticity of variance (M-CEV) model for the asset's price and a power utility over the final wealth for a finite horizon agent. This model was introduced in Heath and Platen (2002) and is a natural extension of the famous CEV model. The model captures the volatility smile effect. That is, it allows non-zero probability of the underlying default (M-CEV process can touch zero while GBM is always positive) and it is analytically tractable. Also this model is applicable to algorithmic trading strategies because M-CEV process has a mean- reversion property for some of the model's parameter. The model obtains a closed form solution in terms of confluent hyper geometric functions, which is where two of the three regular singularities merge into an irregular singularity.

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Recently, more researchers study the utility maximization problem using stochastic control theory.The choice of the utility function is very important. There are some popular utility functions like logarithmic, power and exponential. Obviously, each utility gives a different optimal strategy that maximizes expected utility over the terminal wealth. It is well known that the optimal strategy in the case or a logarithmic utility does not depend on the time to the end of the investing period and the trading rules of an exponential utility investor is not sensitive to the current wealth, Merton (1990). In order to capture time and wealth dependencies we choose a power utility.

The rest of this work is organized as follows; section II reviewed some works done in this field. In section III, we introduce of the financial market and the wealth process of the investor and the formulation of the investor's problem. The derivation of the Hamilton-Jacobi-Bellman (HJB) equation is presented in section IV. Section concludes the paper with findings after which references followed

REVIEW OF RELATED LITERATURE

Optimal investment problem of utility maximization can literally be traced back and viewed from the pioneer work of Merton (1971). Merton's model was used to assess the credit risk of a company debt. Analyst and investors utilize the Merton model to understand how capable a company is at meeting financial obligations, servicing its debt and weighing the general possibility that it will go into credit default. This model was later built out by Fischer Black and Myron Scholes to develop the Black-Scholes (1973) pricing model. In the Merton's (1969) classical portfolio optimization problem, an investor allocates his wealth between one risk asset and one risk-free asset and chooses an optimal consumption rate to maximize total expected discounted utility of consumption.

Chang et al (2014) studied an asset and liability management problem with stochastic interest rate in which interest rate was assumed to follow the affine interest rate model.

Chang et al (2013) in their work "The optimal investment and consumption decisions under the Ho-lee interest rate model" investigates an investment and consumption problem with stochastic interest rate, in which interest rate was assumed to follow the Ho-lee model and be correlated with stock price and derived optimal strategies for power and logarithmic utility function.

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Gao (2009) studied the portfolio problem of a pension fund management in a complete financial market with stochastic interest rate.

Jung et al (2012) "Optimal investment strategies for the HARA utility under the constant elasticity of variance model", gave an explicit expression for the optimal investment strategy, under the constant elasticity of variance model, which maximizes the expected HARA utility of the final value of the surplus at the maturity time. To do this, the corresponding HJB equation was transformed into a linear partial differential equation by applying a Legendre transform and proved that the optimal investment strategy corresponding to the HARA utility function converges as to the one corresponding to the exponential utility function.

Ihedioha (2017) "Investor's Power Utility Optimization with Consumption, Tax, Dividend and Transaction Cost under Constant Elasticity of Variance Model" considered an investor's portfolio where consumption, taxes, transaction costs and dividends are in involved, under constant elasticity of variance model. The stock price is assumed to be governed by constant elasticity of variance model and his goal was to maximize the expected utility of consumption and terminal wealth where the investor has a power utility preference. The application of dynamic programming principles, specifically the maximum principle helped in obtaining the Hamilton-Jacobi-Bellman (HJB) equation for the value function on which elimination of variable dependency was applied to obtain the close form solution of the optimal investment and consumption strategies. It was found that optimal investment on the risky asset is horizon dependent.

Wang et al (2014) studied " The CEV Model and Its Application in a Study of Optimal Investment Strategy", in which they used constant elasticity of variance CEV model to describe the price of the risky asset. Maximizing the expected utility relating to the Hamilton-Jacobi-Bellman (HJB) equation which describes the optimal investment strategies, they obtain a partial differential equation. Applying the Legendre transform, the partial differential equation was transformed to a dual problem which was used obtain an approximation solution and an optimal investment strategy for the exponential utility function.

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Ihedioha (2017) studied "Effect of Correlation of Brownian Motions on an Investor's Optimal Investment and Consumption Decision under Ornstein-Uhlenbeck Model" The aim of his work was to investigate and give a closed form solution to an investment and consumption decision problem where the risk-free asset has a rate of return that is driven by the Ornstein-Uhlenbeck Stochastic interest rate of return model. The maximum principle is applied to obtain the HJB equation for the value function. Owing to the introduction of the consumption factor and the Ornstein-Uhlenbeck Stochastic interest rate of return, the HJB equation derived become more difficult to deal with than the one obtained in literature. The non-linear second-order partial differential equation was transformed into an ordinary differential equation; specifically, the Bernoulli equation, using elimination of dependency on variables which made for the solution obtained.

Wenyuan and Jingtang (2018) "Optimal investment strategies for general utilities under dynamic elasticity of variance models" studied the optimal investment strategies under the dynamic elasticity of variance (DEV) model which maximize the expected utility of terminal wealth. The DEV model is an extension of the constant elasticity of variance model, in which the volatility term is a power function of stock prices with the power being a non-parametric time function. It is not possible to find the explicit solution to the utility maximization problem under the DEV model. In their work, a dual-control Monte-Carlo method is developed to compute the optimal investment strategies for a variety of utility functions, including power, non-hyperbolic absolute risk aversion and symmetric asymptotic hyperbolic absolute risk aversion utilities. Numerical examples show that this dual-control Monte-Carlo method is quite efficient.

Dmitry Muravey worked on an optimization problem assuming the Modified Constant Elasticity of variance (M-CEV) model for the asset's price and a power utility over the final wealth for a finite horizon agent. This model was introduced in Heath and Platen (2002) and is natural extension of the famous CEV mode (see Cox (1975). This model captures the volatility smile effect; allows non-zero probability of the underling's defaults (M-CEV process can touch zero while GBM is always positive); and it is analytically tractable. Also this model is applicable to algorithmic trading strategies because the M-CEV process has mean-reversion property for some of the model's parameters. The time-dependent extension of this model can be found in Linetsky and Carr (2006). For the M-CEV model we obtain a closed form solution in terms of confluent hyper-geometric functions.

METHODOLOGY

In this section, we introduce the idea and concept of the modified constant elasticity of variance model (M-CEV Model) in risk theory; we will be looking at the effects of the Brownian motion, Geometric Brownian motion as the simplest stochastic differential equation, the Constant Elasticity of Variance (CEV), the Ornstein-Uhlenbeck model, the two major tools for studying optimally controlled systems (Dynamic Programming and the maximum principle) involving the Hamiltonian Function with a constant relative risk aversion (CRRA) which in other words is a description of an investor who when faced with two investments with a similar expected return (but different risk) will rather prefer to go for the one with lower risk which is relative to growth objective that may be taken by a prospective investor.

Brownian Motion

Brownian motion is physical phenomena in which a quantity is constantly undergoing small, random fluctuations; it is also a simple continuous stochastic process that is widely used in physics and finance for modeling random behavior that evolves over time. Examples of such behavior are the fluctuations in an asset's price or the random movements of molecule of gas or liquid.

A standard (One-dimensional) Wiener process (also called Brownian motion) is a stochastic process $\{W(t)\}; t \geq 0$ indexed by non-negative real numbers *t* with the following properties.

i. $W(0) = 0$

ii. With probability 1, the function $t \rightarrow w(t)$ is continuous in *t*.

iii. The process $\{W(t)\}; t \geq 0$ has stationary independent increments.

iv. The increment $W(t+s) - W(s)$ has the normal $(0, t)$ distribution.

$N(\mu, \sigma^2)$ denotes the normal distribution with the expected value μ and σ^2. The conditions that it has independent increments means that if; $0 \leq s_1 \leq t_1 \leq s_2 \leq t_2$

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then $W(t_1) - W(s_1)$ *and* $W(t_2) - W(s_2)$ are independent random variables, this is in honour of Norbert Wiener.

Geometric Brownian Motion (GBM)

Geometric Brownian motion is the simplest stochastic differential equation (SDE) from the SDE family. A stochastic process $S(t)$ is said to follow geometric Brownian motion (GBM) if it satisfies the following stochastic differential equation (SDE):

$$dS(t) = \varsigma S(t)dt + \tau S(t)dZ(t),$$

where $Z(t)$ is a Brownian motion(wiener process), ς appreciation rate and τ volatility, are constants or with a deterministic component defined by the function $\varsigma S(t)$, the instantaneous drift is defined by the function $\tau S(t)$, the stochastic differential $dS(t)$ represents an infinitesimal increment.

In other words, Geometric Brownian motion is a continuous – time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion with drift. It is paramount example of a stochastic process satisfying a stochastic differential equation (SDE), in particular, it is used in mathematical finance to model stock prices in the Black – Scholes model.

Constant Elasticity of Variance (CEV) Model

In financial mathematics, the CEV model is a stochastic volatility model whose target is to capture the leverage effect and the stochastic volatility. This model is widely used in the financial industry by most practitioners, especially for modeling equities and commodities.

The CEV model describes a process which evolves according the following stochastic differential equation:

$$dS(t) = S(t)[\varphi dt + \delta S^{\gamma}(t) + dZ(t)] ,$$

where $S(t)$ is the spot price, t is time and φ is a parameter characterizing the drift, δ and γ are other parameters and Z is a Brownian motion (Wiener process).

The notation "$dS(t)$" represents a differential.

The constant parameters φ and γ satisfy the condition $\delta \geq 0$, $\gamma \geq 0$. The parameter γ controls the relationship between volatility and price and is the central feature of the model.

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When $\gamma < 1$ we see the so – called leverage effect, commonly observed in equity markets, where the volatility of a stock increases as its price falls.

Modified Constant Elasticity of Variance (M-CEV)

This model was introduced in Heath and Platen (2002) and is a natural extension of the famous CEV – model (See Cox (1975). We choose this model basically because the model captures the volatility smile and skew effect, it also allows non – zero probability of the underlining's default (M-CEV process can n touch zero while GBM is always positive); and it is analytically tractable. Also the model is applicable to algorithmic trading strategies because the M-CEV process has mean-reversion property for some of the model's parameters.

Consider a simple market consisting of a risk-free bond $B(t)$ and a risky asset (i.e. stock) $S(t)$. The bond and stock prices are driven by SDE:

$$dB(s) = r(s)B(s)d(s); \qquad B(t) = B > 0;$$

$$\frac{dS(s)}{S(s)} = [r(s) - q(s) + \lambda(S(s);\, s)]ds + \sigma(S(s);\, s)dZ(s); \qquad S(t) = S > 0;$$

where $Z(s)$ is a standard Wiener process, $r(s) \geq 0$, $q(s) \geq 0, \sigma(S;\, s) > 0$ and $\lambda(S;\, t) \geq 0$ are the time-dependent risk-free interest rate, the time-dependent dividend yield, the time- and state- dependent instantaneous stock volatility, and the time- and state- dependent default intensity, respectively. The M-CEV model has the following specifications:

$$\sigma(S(s);\, s) = aS^{\gamma},\ \lambda(S;\, s) = b + c\sigma^2\,(S;\, s) = b\, ca^2\, S^{2\gamma}\, q(s) = q,\ r(s) = r$$

$$\alpha = r - q + b;$$

and defined by this corresponded SDE;

$$\frac{dS(s)}{S(s)} = [\alpha + ca^2 S^{2\gamma}]ds + aS^{\gamma} d{Z_1}^{(1)}(s)$$

Let us mention that Heath and Platen considered model above with $c = 1$. The case of $c \neq 1$ is not extension of original M-CEV model because this case can be reduced to the original model by a simple change of measure.

Ornstein – Uhlenbeck Model

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The Ornstein – Uhlenbeck process is one of several approaches used to model (with modifications) interest rates, exchange rates, currency and commodity prices stochastically. An Ornstein – Uhlenbeck process $r(t)$ satisfies the following stochastic differential equation;

$$dr(t) = \theta\big(\mu - r(t)\big)dt + \sigma dZ_2(t),$$

where $\theta > 0$ *and* $\sigma > 0$ are parameters and $Z_2(t)$ denotes the Wiener Process. It is also known as the Vasicek model. The parameter μ represents the equilibrium or mean – value supported by fundamentals, σ the degree of volatility around it caused by stocks and θ the rate by which these stocks dissipate and the variable reverts towards the mean.

The Model and the Model Formulation

Let an investor trade two assets in a financial market, a risky asset (stock) and a risk free asset (bond) that has a rate that is dependent on time, then the dynamics of price of the risk-free asset denoted by $B(t)$ is given by

$$dB(t) = r(t)B(t)dt, \tag{1}$$

from which we get

$$\frac{dB(t)}{B(t)} = r(t)dt \tag{2}$$

The risky asset's price $S(t)$ at time, t, is governed by the modified constant elasticity of variance (M-CEV)

$$dS(t) = S(t)[\alpha + ca^2S^{2\gamma}(t)]dt + aS^{\gamma}(t)dZ_1(t)], \tag{3}$$

Equation (3) can be written as

$$\frac{dS(t)}{S(t)} = [\alpha + ca^2S^{2\gamma}(t)]dt + aS^{\gamma}(t)dZ_1(t)], \tag{4}$$

where $\big(\alpha + ca^2S^{2\gamma}(t)\big)$ is the drift parameter, aS^{γ} is the volatile scale parameter, $Z_1(t)$ is the Brownian motion parameter, γ is the elasticity parameter of the local volatility, $S(t)$ is the prices of the risky asset at time, t.

Let $\pi(t)$ be the amount of money the investor puts in the risky asset at time t, then $[W(t) - \pi(t)]$ is the money amount he invested in the risk-free asset, where W(t) is the total wealth investment

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in both assets. Corresponding to the trading strategy $\pi(t)$, the dynamics of the wealth process follows the stochastic differential equation (S.D.E):

$$dW(t) = \pi(t)\frac{dS(t)}{S(t)} + [W(t) - \pi(t)]\frac{dB(t)}{B(t)}. \tag{5}$$

Applying equation (2), (4) to (7) we get

$$dW(t) = \begin{Bmatrix} [(\alpha - r(t))\pi(t) + r(t)W(t) + ca^2S^{2\gamma}\pi(t)]dt \\ +aS^{\gamma}\pi(t)dZ_1(t) \end{Bmatrix}. \tag{6}$$

The quadratic variation, $<.>$, of equation (6) is

$$< dW(t) >= a^2S^{2\gamma}(t)\pi^2(t)dt \tag{7}$$

where

$$\begin{matrix} dt.dt = dt.dZ_1(t) = 0 \\ dZ_1(t).dZ_1(t) = dt \end{matrix} \tag{8}$$

Suppose the investor has a power utility function $U(W)$, then the investors problem can be written as;

$$G(W,t,T) = Max_{\pi(t)}E[U(W)] \tag{9}$$

subject to,

$$dW(t) = \{[(\alpha - r(t))\pi(t) + r(t)W(t) + ca^2S^{2\gamma}\pi(t)]dt + aS^{\gamma}\pi(t)dZ_1(t)\}.$$

THE OPTIMIZATION

This study assumes that the investor has power utility preference of the form

$$U(W(t)) = \frac{W^q}{q}, q \neq 0, \tag{10}$$

with coefficient of relative risk aversion given as

$$R(W(t)) = \frac{WU''(W)}{U'(W)}, \tag{11}$$

where W(t) is the investor's wealth at time, t.

We consider two cases thus:

Case 1: When the Brownian motions do not correlate: (That is $E[dZ_1.dZ_2] = 0$)

The derivation of the Hamilton-Jacobi-Bellman partial differential equation starts with the Bellman equation: Let investor's value function be

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$$G(W,t,T) = Max_\pi E[G(W',t,T)] \tag{12}$$

where W' denote the investor's wealth process at time $t + \Delta t$, then from equation (12) we have

$$Max_\pi E[G(W',t,T) - G(W,t,T)] = 0 \tag{13}$$

Dividing equation (13) by Δt and taking the limit as $\Delta t \to 0$, we obtain the Bellman equation:

$$Max_\pi \frac{1}{dt} E[dG] = 0 \tag{14}$$

We shall use in (14) the maximum principle which states that

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}ds + \frac{\partial G}{\partial r}dr + \frac{\partial G}{\partial w}dw + \frac{\partial^2 G}{\partial s \partial w}dsdw + \frac{\partial G}{\partial r \partial w}drdw + \frac{\partial^2 G}{\partial s \partial r}dsdr$$
$$+\frac{1}{2}\left[\frac{\partial^2 G}{\partial s^2}(ds)^2 + \frac{\partial^2 G}{\partial r^2}(dr)^2 + \frac{\partial G}{\partial w^2}(dw)^2\right]. \tag{15}$$

But

$$\left.\begin{array}{c} dW(t) = \{[(\alpha - r(t))\pi(t) + r(t)W(t) + ca^2S^{2r}\pi(t)]dt + aS^\gamma\pi(t)dZ_1(t)\} \\ dr(t) = \theta(\mu - r(t)dt + \sigma dZ_2(t) \\ dS(t) = \{S(t)[(\alpha + ca^2S^{2\gamma}(t))dt + aS^\gamma(t)dZ_1(t)]\} \\ < dS(t) > = [S(t)[(\alpha + ca^2S^{2\gamma}(t))dt + aS^\gamma(t)d_{Z_1}(t)]^2 \\ = a^2S^{2(\gamma+1)}(t)dt \\ < dr(t) > = (dr)^2 = \sigma^2 dt \\ < dW(t) > = a^2S^{2\gamma}(t)\pi^2(t)dt = a^2S^{2\gamma}\pi^2 dt \\ (dS(t)dW(t)) = a^2S^{2\gamma}(t)S(t)\pi(t)dt = a^2S^{(2\gamma+1)}\pi dt \\ (dr(t)dW(t)) = 0 \\ (dS(t)dr(t)) = 0 \end{array}\right\} \tag{16}$$

where

$$\left.\begin{array}{l} dt.dt = dt.dZ_1 = dt.dZ_2 = 0 \\ dZ_1.dZ_1 = dZ_2.dZ_2 = dt \\ dZ_1.dZ_2 = 0 \end{array}\right\} \tag{17}$$

Substituting (16) into (15) we obtain

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}\{[S(\alpha + ca^2S^{2\gamma})dt + aS^\gamma dZ]\} + \frac{\partial G}{\partial r}[\theta(\mu - r)dt - \sigma dZ_2] + \frac{\partial G}{\partial w}\{[(\alpha - r)\pi +$$
$$rW + ca^2S^{2\gamma}\pi]dt + aS^\gamma\pi dZ_1\} + \frac{\partial^2 G}{\partial s \partial w}\left[a^2S^{(2\gamma+1)}\pi dt\right] + \frac{\partial^2 G}{\partial r \partial w}[0] + \frac{\partial^2 G}{\partial s \partial r}[0]$$
$$+\frac{1}{2}\left[\frac{\partial^2 G}{\partial s^2}\left(a^2S^{2(\gamma+1)}dt\right) + \frac{\partial^2 G}{\partial r^2}[\sigma^2 dt] + \frac{\partial^2 G}{\partial w^2}(a^2S^{2\gamma}\pi^2 dt)\right] \tag{18}$$

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Using (18) in (14), we get

$$Max_{\pi} \frac{1}{dt} E \left\{ \frac{\partial G}{\partial t} dt + \frac{\partial G}{\partial s} \{S[(\alpha + ca^2 S^{2\gamma})dt + aS^{\gamma} dZ_1]\} + \frac{\partial G}{\partial r} [\theta(u-r)dt - \sigma dZ_2] + \frac{\partial G}{\partial w} \{[(\alpha - r)\pi + rW + ca^2 S^{2\gamma}\pi]dt + aS^{\gamma}\pi dZ_1\} + \frac{\partial^2 G}{\partial s \partial w} (a^2 S^{(2\gamma+1)} \pi dt) + -\frac{1}{2} \left[\frac{\partial^2 G}{\partial S^2} \left(a^2 S^{2(\gamma+1)} dt\right) + \frac{\partial^2 G}{\partial r^2} (\sigma^2 dt) + \frac{\partial^2 G}{\partial w^2} (a^2 S^{2\gamma} \pi^2 dt) \right] \right\} = 0 \quad (19)$$

Rewriting (20), we get;

$$G_t + G_s S(\alpha + Ca^2 S^{2\gamma}) + G_r\big(\theta(u-r)\big) + G_w(\alpha - r)\pi + rW + ca^2 S^{2\gamma}\pi + G_{sw}\left(a^2 S^{(2\gamma+1)}\pi\right) + G_{ss}\frac{a^2 S^{2(\gamma+1)}}{2} + G_{rr}\frac{(\sigma^2)}{2} + G_{ww}\frac{(a^2 S^{2\gamma}\pi^2)}{2} = 0 \quad (21)$$

Where;

$$E\,[dZ_1 . dZ_2] = 0 \quad (22)$$

Differentiating (21) with respect to π gives,

$$G_w[(\alpha - r) + ca^2 S^{2\gamma}] + G_{sw}\left(a^2 S^{(2\gamma+1)}\right) + G_{ww}(a^2 S^{2\gamma}\pi) = 0 \quad (23)$$

Making π the subject of (23),we get

$$\pi = -\left[\frac{(\alpha + Ca^2 S^{2\gamma} - r)G_w}{(a^2 S^{2\gamma})G_{ww}} + \frac{(a^2 S^{(2\gamma+1)})G_{sw}}{(a^2 S^{2\gamma})G_{ww}}\right] \quad (24)$$

Let $G\,(t, s, r, w) = \frac{W^q}{q} h(t, s, r)$ (25)

Be a solution to the HJB equation in (21) above, we obtain the following

$G_t = \frac{W^q}{q} h_t$, $G_s = \frac{W^q}{q} h_s$, $G_r = \frac{W^q}{q} h_r$, $G_w = W^{q-1}\, h$,

$G_{sw} = W^{q-1}\, h_s$, $G_{rw} = W^{q-1}\, h_r$,

$G_{sr} = W^{q-1}\, h_r$, $G_{ss} = W^{q-1}\, h_{ss}$, $G_{rr} = W^{q-1}\, h_{rr}$,

$G_{ww} = q - 1W^{q-2}h.$ (26)

Again applying (25) in (24) and simplifying yields;

$$= -\left[\frac{(\alpha + ca^2 S^{2\gamma} - r)W}{(a^2 S^{2\gamma})(q-1)} + \frac{SWh_S}{(q-1)h}\right] \quad (27)$$

To eliminate dependency on S, we let

$$h(t, r, s) = y(t, r)\frac{S^q}{q} \quad (28)$$

Such that at terminal time T,

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$$\frac{q}{s^q} = y(T,r) \tag{29}$$

From (28), we have;

$$h_s = S^{q-1}y(t,r), \quad h_r = \frac{S^q}{q}y_r \tag{30}$$

Using (28) in (27)

$$\pi = -\left[\frac{(\alpha+ca^2S^{2\gamma}-r)}{(q-1)a^2S^{2\gamma}} + \frac{q}{(q-1)}\right]W \tag{31}$$

Case 2: When the Brownian motion correlate:$(\boldsymbol{That\ is\ E}\ [\boldsymbol{dZ_1.dZ_2}] = \boldsymbol{\rho dt})$

The equation (17) becomes

$$\left.\begin{array}{c} dW(t) = \{[(\alpha - r(t)\pi(t) + r(t)W(t) + ca^2S^{2\gamma}\pi(t)]dt + aS^\gamma\pi(t)dZ_1(t)\} \\ dr(t) = \theta\big(\mu - r(t)\big)dt + \sigma dZ_2(t) \\ dS(t) = \{S(t)[(\alpha + ca^2S^{2\gamma}(t)dt + aS^\gamma(t)dZ_1(t) \\ < dS(t) > = \{S(t)[(\alpha + ca^2S^{2\gamma}(t)dt + aS^\gamma(t)dZ_1(t)\}^2 \\ = a^2S^{2(\gamma+1)}\,(t)dt \\ = a^2S^{2(\gamma+1)}\,dt \\ < dr(t) > = (dr)^2 = \sigma^2dt \\ < dW(t) > = a^2S^{2\gamma}(t)\pi^2(t)dt \\ = a^2S^{2\gamma}\pi^2dt \\ (dS(t)dW(t)) = a^2S^{2\gamma}(t)S(t)\pi(t)dt = a^2S^{(2\gamma+1)}\pi dt \\ (dr(t)dW(t)) = \rho\sigma aS^\gamma\pi dt \\ (dS(t)dr(t)) = \rho\sigma aS^{(\gamma+1)}dt \end{array}\right\} \tag{32}$$

where

$$\left.\begin{array}{c} dt.dt = dt.dz_1 = dt.dz_2 = 0 \\ dz_1.dz_1 = dz_2.dz_2 = dt \\ dz_1.dz_2 = \rho dt \end{array}\right\} \tag{33}$$

Again, substituting (32) into (16) we obtain

$$dG = \frac{\partial G}{\partial t}dt + \frac{\partial G}{\partial s}[s(\alpha + ca^2S^{2\gamma})dt + aS^\gamma dz_1] + \frac{\partial G}{\partial r}[\theta(\mu - r)dt + \sigma dz_2]\frac{\partial G}{\partial w}\{\big[\big(\alpha - r(t)\big)\pi + rW + ca^2S^{2\gamma}\pi\big]dt + aS^\gamma\pi dz_2\} + \frac{\partial^2 G}{\partial s\partial w}\big[a^2S^{(2\gamma+1)}\pi dt\big] + \frac{\partial^2 G}{\partial r\partial w}[\rho\sigma aS^\gamma\pi dt] + \frac{\partial^2 G}{\partial s\partial r}[\rho\sigma aS^{(\gamma+1)}\pi dt] + \frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}[a^2S^{2(\gamma+1)}dt] + \frac{\partial^2 G}{\partial r^2}[\sigma^2dt] + \frac{\partial G}{\partial w^2}[a^2S^{2\gamma}\pi^2dt]\right]$$

(34)

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Using (34) in (15) we obtain;

$$Max_{\pi}\frac{1}{dt}\,E\left\{\frac{\partial G}{\partial t}dt+\frac{\partial G}{\partial s}\left[s(\alpha+ca^{S^{2\gamma}2})dt+aS^{\gamma}dz_1\right]+\frac{\partial G}{\partial r}[\theta(\mu-r)dt+\ \sigma dz_2]\frac{dG}{dw}\{[(\alpha-r]\pi+rW+ca^2S^{2\gamma}\pi]dt+aS^{\gamma}\pi dz_2\}+\frac{\partial^2 G}{\partial s\partial w}\left[a^2S^{(2\gamma+1)}\pi dt\right]+\frac{1}{2}\left[\frac{\partial^2 G}{\partial S^2}[a^2S^{2(\gamma+1)}dt]+\frac{\partial^2 G}{\partial r^2}[\sigma^2 dt]+\frac{\partial^2 G}{\partial w^2}[a^2S^{2\gamma}\pi^2 dt]\right]\right\}=0$$

(35)

We rewrite (35) to obtain

$$G_t+G_s[S(\alpha+ca^2S^{2\gamma})]+G_r[\theta(\mu-r)]+G_w[(\alpha-r\pi+rw+ca^2S^{2\gamma}\pi)]+G_{sw}\left(a^2S^{(2\gamma+1)}\pi\right)+G_{rw}(\rho\sigma aS^{\gamma}\pi)+G_{sr}\left(\rho\sigma aS^{(\gamma+1)}\right)+\frac{G_{ss}a^2S^{2(\gamma+1)}}{2}+\frac{G_{rr\sigma^2}}{2}+G_{ww}\frac{a^2S^{2\gamma}\pi^2}{2}=0$$

(36)

where;

$$E\,[dZ_1.dZ_2]=0 \qquad (37)$$

The differentiation of (36) with respect to π gives

$$G_w(\alpha-r)+ca^2S^{2\gamma}+G_{sw}\left(a^2S^{(2\gamma+1)}\right)+G_{rw}(\rho\sigma aS^{\gamma})+G_{ww}(a^2S^{2\gamma}\pi)=0 \quad (38)$$

Making π the subject of (38)

$$\pi=-\left[\frac{[(\alpha-r)+ca^2S^{2\gamma}]G_w}{(a^2S^{2\gamma})G_{ww}}+\frac{(\rho\sigma aS^{\gamma})G_{rw}}{(a^2S^{2\gamma})G_{ww}}+\frac{(a^2S^{(2\gamma+1)})G_{sw}}{(a^2S^{2\gamma})G_{ww}}\right] \qquad (39)$$

Applying (25) in (39)

$$\pi=-\left[\frac{(\alpha+ca^2S^{2\gamma}-r)W}{(q-1)(a^2S^{2\gamma})}+\frac{\rho\sigma Wh_r}{(q-1)(aS^{\gamma})h}+\frac{SWh_S}{(q-1)h}\right] \qquad (40)$$

Again applying (30) to (40), we get;

$$\pi=-\left[\frac{(\alpha+ca^2S^{2\gamma}-r)W}{(q-1)(a^2S^{2\gamma})}+\frac{\rho\sigma W\frac{S^q}{q}y_r}{(q-1)(aS^{2\gamma})\frac{S^q}{q}y(t,r)}+\frac{SWS^{q-1}\,y(t,r)}{(q-1)\frac{S^q}{q}y(t,r)}\right] \qquad (41)$$

Equation (41) simplifies to

$$\pi=-\left[\frac{(\alpha+ca^2S^{2\gamma}-r)W}{(q-1)a^2S^{2\gamma}}+\frac{\rho\sigma Wy_r}{(q-1)(aS^{2\gamma})y}+\frac{qW}{(q-1)}\right] \qquad (42)$$

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To eliminate the dependency on r we let

$$y(r,t) = \frac{r^q}{q} J(t) \quad (43)$$

Such that at terminal time T,

$$J(t) = \frac{q^2}{(rS)^q} \quad (44)$$

From (43), we get

$$y_r = r^{q-1} J \quad (45)$$

Substituting (45) in (42), we obtain

$$\pi = -\left[\frac{(\alpha+ca^2S^{2\gamma}-r)W}{(q-1)(a^2S^{2\gamma})} + \frac{\rho\sigma W r^{q-1} J}{(q-1)(aS^{\gamma})\frac{r^q}{q} J} + \frac{qW}{(q-1)}\right] \quad (46)$$

Equation (46) simplifies to

$$\pi_{Cor} = -\left[\frac{(\alpha+ca^2S^{2\gamma}-r)}{(q-1)(a^2S^{2\gamma})} + \frac{\rho\sigma q}{(q-1)arS^{\gamma}} + \frac{q}{(q-1)}\right] W \quad (47)$$

Effects of the Correlation of the Brownian Motions

The optimal investment strategy when the Brownian motions do not correlate is given by;

$$\pi_{no\ Cor} = -\left[\frac{(\alpha+ca^2S^{2\gamma}-r)}{(q-1)a^2S^{2\gamma}} + \frac{q}{(q-1)}\right] W \quad (48)$$

And when the Brownian motions correlate by;

$$\pi_{Cor} = -\left[\frac{(\alpha+ca^2S^{2\gamma}-r)}{(q-1)(a^2S^{2\gamma})} + \frac{\rho\sigma q}{(q-1)arS^{\gamma}} + \frac{q}{(q-1)}\right] W \quad (49)$$

Therefore;

$$\pi_{Cor} = \pi_{no\ Cor} - \left[\frac{\rho\sigma q}{(q-1)arS^{\gamma}}\right] W \quad (50)$$

It can be seen from equation (50), that when $\rho = 0$, the case of non-correlation of the Brownian motions is obtained.

This study assumes four cases for the correlation of the Brownian motion namely;

CASE 1: When the correlation is unity, say $\rho = 1$

We obtain

$$\pi_{Cor} = \pi_{no\ Cor} - \left[\frac{\sigma q}{(q-1)arS^{\gamma}}\right] W \quad (51)$$

(That is, when $\rho = 1$, the investor's optimal strategy when the Brownian motions correlate is greater than the investor's optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{\sigma q}{(q-1)ars^{\gamma}}\right]$ of the total wealth).

CASE 2: When the correlation is negative, say $\rho = -K$,

We obtain;

$$\pi_{Cor} = \pi_{no\ Cor} + \left[\frac{\sigma q}{(q-1)ars^{\gamma}}\right] W \tag{52}$$

(That is, when $\rho = -K$, the investor's optimal investment strategy when the Brownian motions correlate is less than the investor's optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{k\sigma q}{(q-1)ars^{\gamma}}\right]$of the total wealth).

CASE 3: When the correlation is positive, say $\rho = K$, that is; $\rho = K$

We obtain

$$\pi_{Cor} = \pi_{no\ Cor} - \left[\frac{k\sigma q}{(q-1)ars^{\gamma}}\right] W \tag{53}$$

(That is, when $\rho = K$ the investor's optimal investment strategy when Brownian motions correlate is greater than the investor's optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{k\sigma q}{(q-1)ars^{\gamma}}\right]$ of that total wealth).

CASE 4: When the correlation is equal zero, say $\rho = 0$,

We obtain;

$$\pi_{Cor} = \pi_{no\ Cor} - \left[\frac{k\sigma q}{(q-1)ars^{\gamma}}\right] W \tag{54}$$

$$\pi_{Cor} = \pi_{no\ Cor}$$

(That is, when $\rho = 0$ the investor's optimal investment strategy when Brownian motions correlate is equal to the investor's optimal investment strategy when Brownian motions do not correlate).

CONCLUSION

This work investigated on investor's investment strategy problem. The work starts with introduction in chapter (1). Chapter (2) various works done in this field the introduction of the financial market is done and the wealth process established in chapter (3). It assumed the stock

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price followed the modified elasticity of variance (M-CEV) model a natural extension of the geometric Brownian motion (GBM). In chapter (4) the derivation of the Hamilton-Jacobi-Bellman (HJB) is done with the help of the dynamic programming principles specially the maximum principle and the conjectures on elimination of variables obtained close-form solutions to the optimal preference. Introduction of the unity function and explicit solution of the investor optimal strategy problem obtained when the Brownian motions do not correlate and when the Brownian motions do not correlate and when the Brownian motions do correlate is seen in this chapter, and when studied in four different cases.

Finally in chapter (5), we conclude that the investor optimal investment strategy when the Brownian motions correlate is less than the investor optimal investment strategy when the Brownian motions do not correlate by a fraction $\left[\frac{\rho\sigma q}{(q-1)ars^{\gamma}}\right]$ of the total wealth.

REFERENCES

Chang H, Chang K, and Lu J. M. (2014) portfolio selection with liability and affine interest rate in the HARA utility framework, abstract and applied analysis. 2014, pp. 1-12.

Chang H., Rong X. M. and Zhao H. (2013) optimal investment and consumption decisions under the Ho-Lee interest rate model. WSEAS Transactions on Mathematics. 12, 2013, pp. 1065-1075.

International Journal of Physics and Mathematical Sciences
2020, Vol. 1, Issue 1, pp. 1-16

www.ecejournals.org

Gao J. (2009): Optimal investment strategy for annuity contracts under the constant elasticity of variance model. Insurance: Mathematics and Economics, Vol. 45, No.l, 2009, pp. 9-18.

Heath. D and Platen. E (2002). Consistent pricing and hedging for a modified constant elasticity of variance model. Quantitative Finance 5,459-467.

Ihedioha S. A (2017). Investor's Power Utility Optimization with Consumption, Tax, Dividend and Transaction Cost Under Constant Elasticity of Variance Model. Asian Research journal of Mathematics, vol 4(2); pp. 1-12, 2017; Article no. ARJOM. 33425. ISSN: 2456-477X.

Ihedioha S. A., Oruh B, I. & Osu B. O. (2017): Effects of Correlation of Brownian motions on an investor's Optimal Investment and Consumption Decision under Ornstein-Uhlenbeck Model, Academic Journal of Applied Mathematics Sciences, Academic Research Publishing Group, vol 3(6), pages 52-61, 06-2017. Handle: RePEc: arp: ajaoms:2017: p:52-61.

Jung K. J. H., (2012). Optimal investment strategies for the HARA utility under the constant elasticity of variance model Insurance: Mathematics and Economics, Vol. 51, No. 3, pp.

Muravey .D (2018). Optimal investment problem with M-CEV model: closed form solution and applications to the algorithmic trading 10 July.

Wang A., (2014). CEV model and its application a study of optimal investment strategy. Mathematical problems in engineering.

Wenyuan L. and Jingtang M., (2018). Optimal investment strategies for general utilities under dynamic elasticity of variance models. Quantitative finance 18(8), 1379-1388.

Zhao H. Rong X. (2012): Portfolio selection problem with multiple risky assets under the constant elasticity of variance model Insurance: Mathematics and Economics, Elsevier; vol. 50(1), pages 179-190.